

A self-referred approach to lacunarity

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This letter describes an approach to lacunarity which adopts the pattern under analysis as the reference for the sliding window procedure. The superiority of such a scheme with respect to more traditional methodologies, especially when dealing with finite-size objects, is established and illustrated through applications to DLA pattern characterization. It is also shown that, given the enhanced accuracy and sensitivity of this scheme, the shape of the window becomes an important parameter, with advantage for circular windows.

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Several interesting natural and abstract phenomena and structures are characterized by intricate geometries whose properties can vary along space and/or time [1, 2, 3, 4, 5]. Chaotic dynamics, for instance, is known to be organized in terms of fractal attractors [6], which are characterized by self-similarity or self-affinity over spatial scales. Given that great part of the systems exhibiting particularly interesting behavior involves such complex geometrical organizations, it becomes important to have proper and effective measurements allowing the objective and meaningful quantification of specific geometrical features, such as regularity, density, self-similarity and translational invariance. One important point to be highlighted at the outset is the fact that such measurements are almost invariably *incomplete* or *degenerated*, as a consequence of the mapping from a higher dimensional space, where the structures ‘live’, into a lower dimensional space. Therefore, while it is often unfeasible to incorporate all information into geometrical measurements, they must be capable of expressing the features of particular interest with respect to each specific application. For instance, the characterization of the distribution of empty space of different sizes and shapes is a major factor to be considered while specifying the mechanical properties of a metal bar for example, establishing an intrinsic relationship between topological/geometrical properties and physical strength that can be to some extent captured by its porosity value in case a parsimonious description is needed. By providing accurate and meaningful information about the specific geometrical properties of interest, proper measurements of complex structures allow the construction of statistical models of the analyzed objects and the identification of prototypes, as well as the taxonomic organization of several types of patterns. Such possibilities are important not only for practical applications, but also for theoretical studies aimed at investigating critical phenomena and universality [7].

One of the best known measurements of complex structures is the *fractal dimension*, introduced by B. Mandelbrot, see [8]. Although several alternative definitions of such a measurement have been available for a long time (e.g. [9, 10]), they all assume self-similar (or self-affine)

symmetries while sharing the ability to quantify the spatial ‘complexity’ of given patterns. Although powerful and widely used, the fractal dimension is inherently a degenerated feature, implying an infinite amount of distinct fractals to be mapped into the same fractal dimension. The concept of *lacunarity* [7, 8, 11] has been introduced and used as a means to complement the quantification of complex geometries provided by the fractal dimension. In particular, the lacunarity quantifies the degree of *translational invariance* of the analyzed objects, with low values of lacunarity indicating high levels of such an invariance. A particularly representative illustration of the potential of the combined use of the fractal dimension and lacunarity is related to the characterization of DLA structures which, by being organized around the initial ‘seed’, tend to exhibit distinct geometrical properties around that seed and also at the DLA boundaries [12].

Despite the promising potential of the lacunarity as a measurement of complex patterns, some remaining intrinsic difficulties have conspired to impinge some degree of arbitrariness, constraining its applications. Of special importance is the lack of a proper procedure to treat finite-size objects. Indeed, while lacunarity and fractal dimension are often considered for the characterization of infinite/periodical structures, the treatment of finite and isolated objects, implied by many relevant natural situations, has received relatively little attention in the literature.

The current work investigates the use of the own analyzed pattern as the reference for the windowing procedure underlying lacunarity estimation. Although such an approach has been considered previously [11], the restricted conditions adopted for its validation (Cantor dust) implied its premature dismissal. An interesting informal interpretation of this approach is to understand the structure of interest as being measured by an inhabitant of the object who, therefore, can only sample a circular region around each of its positions. We show in the following that this self-reference windowing system does allow a series of superior features, including enhanced objectivity, accuracy and sensitivity, also implying the shape of the sliding window to become critical for proper

operation. It is shown that such an alternative procedure allows the additional bonus of enhanced computational speed.

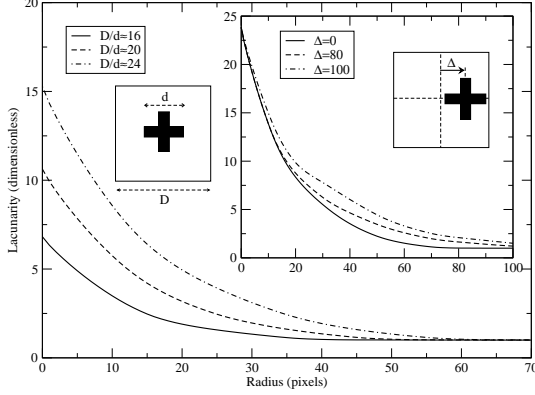


FIG. 1: Some aspects constraining the practical use of the standard implementation of the lacunarity concept for finite objects characterization.

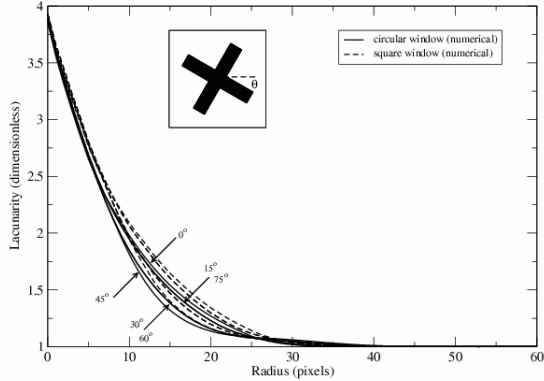


FIG. 2: Inconsistency under object rotation of the usual procedure for measuring the lacunarity concept.

I. METHODOLOGY

One of the most traditional approaches to estimate the lacunarity of a set of objects is known as the gliding-box algorithm, (e.g. [11]). Provided the set under analysis is mapped into a square lattice of dimension $L \times L$, henceforth called the *workspace*, a window of side l is made to slide through the entire lattice while the number of pixels which falls inside it is determined and recorded.

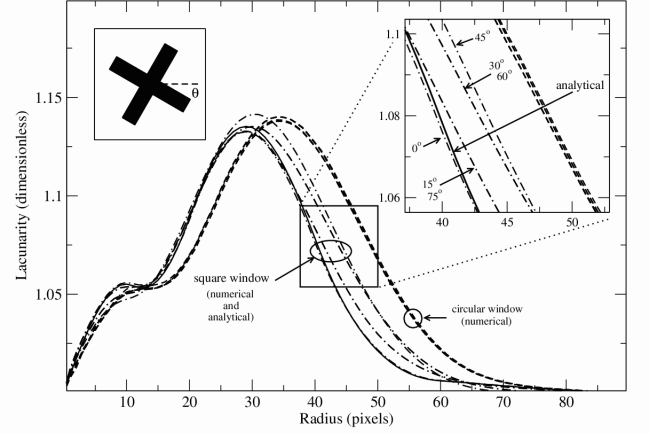


FIG. 3: Stability of the proposed lacunarity measurement under rotation of the object under analysis.

Let $n(s, l)$ be the number of such boxes which contain s pixels and $N(l)$ be the total number of boxes of size l . The probability of finding a box of size l with s pixels is given by $Q(s, l) = n(s, l)/N(l)$, and the lacunarity $\Lambda(l)$ of such a pixel distribution can be simply expressed as

$$\Lambda(l) = \frac{\sum s^2 Q(s, l)}{[\sum s Q(s, l)]^2} = \frac{\sigma_s^2(l)}{\bar{s}^2(l)} + 1 \quad (1)$$

Although popular, such a procedure involves some arbitrariness related to the difficulty to choose the several involved parameters such as the position and size of the workspace and the shape of the sliding window. Figure 1 illustrates the results of applying the sliding approach considering a square window to the inlaid cross. The outset presents three lacunarity signatures obtained for three distinct ratios between the working space and object sizes. It is clear from these curves that the choice of proportionality ratio can have great effect in defining the lacunarity values. The inset curves were obtained for a fixed working space size, but with the object (a cross) placed at different relative positions. A strong variation of the obtained lacunarity values was again observed, indicating arbitrariness also regarding the object position. Therefore, the large variations implied by the above arbitrary choices undermine the potential of the lacunarity as a sensitive measurement of the spatial distribution of the analyzed finite structures. The orientation of the object under analysis represents an additional arbitrary aspect of the traditional sliding-window approach.

The arbitrariness identified above can be completely removed by the use of the structure under analysis as the reference for placing the sliding windows. In other words, the window is placed at each of the points of that structure, eliminating the influence of the workspace, which

can now be objectively defined by the maximum sliding-window size and the structure under analysis. The remaining parameters are therefore reduced to the shape of the sliding-window and the spatial-scale interval of the analysis (i.e. the range of window sizes), accounting for enhanced objectiveness of the whole approach.

Figures 2 and 3 illustrate the stability of the traditional and proposed lacunarity functionals regarding rotation. Figure 2 shows the traditional lacunarity curves obtained for rotations of the considered pattern considering square and circular windows. A substantial variation is observed for both types of windows. Figure 3 gives the self-referred lacunarity of the object shown on the upper left for diverse rotation angles, two sliding window shapes, as well as the respective analytical result. In the main graph we can easily spot two main groups: one associated with the square sliding window and another, more tightly grouped, associated with the circular window. Among the more widespread group, one can see a solid line representing the analytical calculation expected for the self-reference method, which matches precisely the numerically evaluated curves. The inset provides a zoomed view of the variance implied by the use of a square sliding window, which is particularly critical if rotational invariance is required.

II. RESULTS

An important feature of many quasi self-similar shapes is the existence of a descriptor, such as the fractal dimension, which can provide a characteristic signature for the shape regardless of the number of aggregated particles. The lacunarity represents one such a descriptor which has been proposed in order to complement the fractal characterization and exhibits a ‘convergent behavior’ as the number of particles reaches a critical value [2]. Figure 4 illustrates such a property with respect to the standard procedure for DLA generation [13]. The behavior of the maximum of such curves suggests itself as a possible measurement for characterizing the whole sequence of produced individual shapes.

In order to investigate the potential of the self-referred lacunarity approach for pattern discrimination, an experiment has been carried out in which two differently grown sets of DLA structures [13, 14] with 30 samples each, are analyzed by the self-referred approach described in this paper. The results of this experiment are summarized in Figure 5, which shows two clearly separated clusters (as illustrated by the straight dashed frontier), with some overlap at their borders. Such an overlap is a consequence of some degree of similarity between the two types of structures, which was detected by the considered measurement. It is observed that, out of the two considered measurements, the highest lacunarity value (represented along the x -axis) contributed more effectively to the separation between the two classes of objects.

Another important issue is related to the sensitivity of

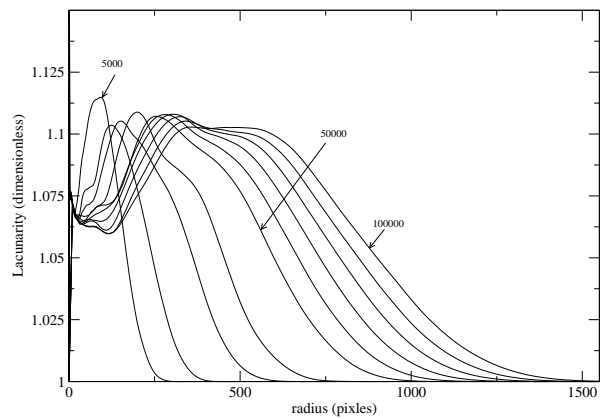


FIG. 4: The effect of the number of aggregated particles on the self-referred lacunarity value. As the number of particles grows, the lacunarity maximum tends to a standstill, suggesting itself as a prominent feature for morphological characterization of such complex shapes, as exemplified by Figure 5.

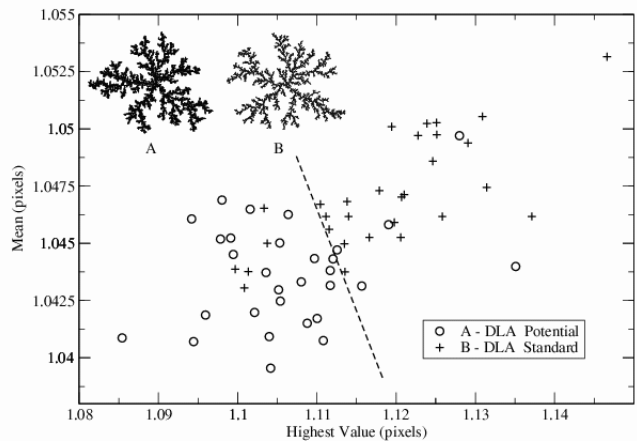


FIG. 5: A scatter plot defined by functionals extracted from the lacunarity curve, namely the global standard deviation and the local maximum value. The plot shows some amount of similarity among the two differently grown types of DLAs.

this new approach to small variations of the object. We consider this important perspective through an experiment where the object is perturbed by increasing Poisson noise. The outcome of such a study is presented in figure 6, which shows the self-referred lacunarity curves obtained for several levels of noise, quantified by the respective Poisson rates given by the respective legend. While

the traditional lacunarity, shown in the outset graph, is characterized by maximum relative variation of 0.73 against 0.13. Such a result suggests that the self-referred method present enhanced robustness when compared to the traditional lacunarity.

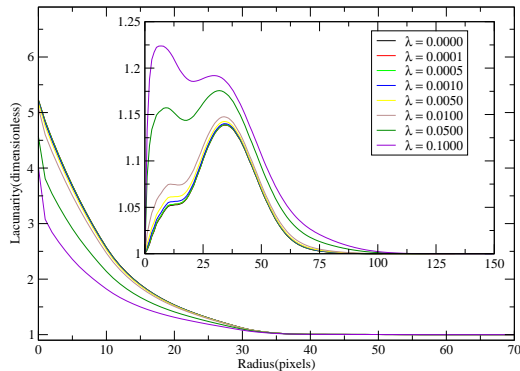


FIG. 6: The resilience of the proposed lacunarity against perturbation by Poisson noise with varying number density.

III. COMMENTS AND CONCLUSIONS

While the traditional approach to lacunarity estimation involves sliding a box throughout the space where the structure under analysis is contained, the application of such a procedure implies substantial arbitrariness when applied to general shapes and sets of objects characterized by finite size. Of particular importance is the

fact that there is no established criterion for defining the positions of the sliding box along the space under analysis, so that different implementations will often converge to different results. We have shown that the adoption of the own objects under analysis as the reference for positioning of the sliding window provides not only a fully objective procedure for lacunarity estimation, but also enhances its potential for discriminating between different classes of patterns. Such effects have been demonstrated with respect to the important problem of DLA pattern formation and analysis. In addition, the stability of the self-referred approach has been investigated with respect to Poisson perturbations, suggesting good robustness. Moreover, the enhanced signature provided by the object-referred framework considered in this article makes the choice of the window geometry an important issue. In particular, we have shown that circular (spherical) windows provide superior properties when used for self-referred lacunarity estimation by promoting the isotropy of the analysis. An additional advantage allowed by the considered lacunarity definition is its substantially reduced demand for computational resources. As the sliding window is constrained to the object under analysis, the total of integrations along the window is reduced from a large area around the object to its own area, which often imply savings of an order of magnitude.

Acknowledgments

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